J Stat Phys (2010) 139: 108–121
DOI 10.1007/s10955-010-9929-4

Stationarity of SLE

Antti Kemppainen

Received: 30 October 2009 / Accepted: 27 January 2010 / Published online: 19 February 2010

Abstract A new method to study a stopped hull of $\mathrm{SLE}_\kappa(\rho)$ is presented. In this approach, the law of the conformal map associated to the hull is invariant under a SLE induced flow. The full trace of a chordal SLE_κ can be studied using this approach. Some example calculations are presented.

Keywords Schramm-Loewner evolutions · Stationarity · Reversibility · Random curves

1 Introduction

Schramm–Loewner evolution (SLE) was introduced by Oded Schramm [8]. SLEs are random curves in the plane. There are many variants of SLE, but the local properties of the random curve are determined by a single parameter $\kappa \geq 0$. SLEs are characterized by conformal invariance and the domain Markov property. The scaling limits of two-dimensional statistical physics models at criticality are believed to be conformally invariant. For this reason the scaling limit of a curve emerging from such a model has to be SLE_κ for some $\kappa \geq 0$. The parameter κ describes the universality class of the model.

A chordal SLE is a random curve in a simply connected domain connecting two boundary points. In Sect. 2, we will define the chordal SLE in more detail. The chordal SLE is *stationary* in the sense that given the process up to a time t the law of K_{t+s} is such that $g_t(K_{t+s} \setminus K_t) - X_t$ and K_s have the same law, where $(g_t)_{t\geq 0}$ is the collection of conformal mappings satisfying the Loewner equation, $(K_t)_{t\geq 0}$ denotes the corresponding collection of subsets of the upper half-plane $\mathbb{H}$, and $(X_t)_{t\geq 0}$ is the driving process.

This work was financially supported by Academy of Finland and by Finnish Academy of Science and Letters, Vilho, Yrjö and Kalle Väisälä Foundation.

A. Kemppainen (✉)
Department of Mathematics and Statistics, University of Helsinki, P.O. Box 68, 00014 Helsinki, Finland
e-mail: antti.h.kemppainen@helsinki.fi

Present address:
A. Kemppainen
Université Paris-Sud 11, Orsay Cedex, France

$\mathrm{SLE}_\kappa(\rho)$-processes, $\kappa \geq 0$ and $\rho \in \mathbb{R}$, are generalizations of the chordal SLE_κ. When $\rho = 0$ this reduces to the chordal case: $\mathrm{SLE}_\kappa(0)$ is the chordal SLE_κ. The definition of $\mathrm{SLE}_\kappa(\rho)$ requires two marked points. If X_t is the driving process of a $\mathrm{SLE}_\kappa(\rho)$ and the other marked point is Y_t, then for a range of the parameter values the hitting time $\tau = \inf\{t \geq 0 : |Y_s - X_s| \to 0 \text{ as } s \nearrow t\}$ is almost surely finite. The stopped hull K_τ is a interesting object in many ways. For example, $\mathrm{SLE}_\kappa(\kappa - 6)$ is a coordinate transformation of the chordal SLE_κ and hence K_τ describes the full SLE_κ trace seen from a fixed point in the real axis.

The novel result of this paper is a formulation of the *stationarity* of $\mathrm{SLE}_\kappa(\rho)$ in Theorem 1 so that K_τ is invariant under the flow which the $\mathrm{SLE}_\kappa(\rho)$ induces. In this approach, the $\mathrm{SLE}_\kappa(\rho)$ is run for a time $t > 0$, then this beginning is erased, and scaling and translation are used to map the beginning and end points X_t and Y_t back to the initial values X_0 and Y_0. By the property stated in Theorem 1, $(g_t(K_\tau \setminus K_t) - \beta_t)/\alpha_t$ has the same law as K_τ, where α_t and β_t are the appropriate scaling and translation factors.

Theorem 1 enables us to calculate quantities related to K_τ such as the moments $\mathbb{E}[\prod_{j=1}^n a_{k_j}]$ of the coefficient of the expansion $G(z) = g_\tau(z) = z + \sum_j a_j z^{-j}$. The driving function and the coefficients of the Loewner map can be viewed as the "state of SLE" and they form the SLE data. The stationarity gives a new way to calculate the distribution functions or the expected values of the SLE data. This is related to the approach in [4], although the work of this paper was done before that paper.

In Sects. 3.4 and 3.5, an approach for the reversibility of the chordal SLE is proposed, and for $\rho = \kappa - 6$, the general form of $\mathbb{E}[\prod_{j=1}^n a_{k_j}]$ as a function of κ is derived using the reversibility. The reversibility was recently proven to hold for chordal SLE_κ, $\kappa \in [0, 4]$ by Dapeng Zhan [12]. It is a property of SLE that states that if the roles of the beginning and end points are changed, then the law of the random curve remains the same.

In Sect. 3.6, moments of the form $\mathbb{E}[a_1^n]$ and $\mathbb{E}[a_1^n a_2^m]$ are calculated. In Sect. 3.7, the method is used to derive the distribution of a_1.

2 SLE and Schramm's Principle

2.1 Chordal SLE

One natural choice for a simply connected domain in the complex plane having two marked boundary points is the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$. The marked points are 0 and ∞. The triplet $(\mathbb{H}, 0, \infty)$ is preserved by the family of mappings $z \mapsto \lambda z$, $\lambda > 0$. The Schwarz lemma shows that these are the only conformal mappings with this property.

A subset $K \subset \mathbb{H}$ is a *hull* if $K = \mathbb{H} \cap \overline{K}$, K is bounded and $\mathbb{H} \setminus K$ is simply connected. If $\gamma : [0, T] \to \mathbb{C}$ is a simple curve such that $\gamma(0) \in \mathbb{R}$ and $\gamma(0, T] \subset \mathbb{H}$, then $K_t = \gamma(0, t]$ is a hull for each $t \in [0, T]$. In this case the family $(K_t)_{t \in [0,T]}$ is growing in the sense that $K_t \subsetneq K_s$ when $0 \leq t < s \leq T$.

Let $(K_t)_{t \geq 0}$ be a growing family of hulls and g_t be the conformal mapping from $\mathbb{H} \setminus K_t$ onto $\mathbb{H}$ that is normalized by $g_t(z) = z + o(1)$ as $z \to \infty$. This normalization makes g_t unique. If $K_0 = \emptyset$ and $(K_t)_{t \geq 0}$ grows continuously in a quite natural sense, we can reparameterize K_t so that $g_t(z) = z + 2t/z + \cdots$ at infinity.

If $(K_t)_{t \geq 0}$ grows locally in the sense of Theorem 2.6 of [6] then the family of mappings $(g_t)_{t \geq 0}$ satisfies the upper half-plane Loewner equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - X_t} \tag{1}$$

where $X_t \in \mathbb{R}$ is called the driving function (process) of K_t. In fact X_t is the image of the point where K_t is growing under the mapping g_t, that is $X_t = \bigcap_{s>t} \overline{g_t(K_s \setminus K_t)}$. Note that the family of hulls given by a simple curve is growing locally.

Consider now a collection of probability measures $(\mu_{\Omega,a,b})$ such that $\mu_{\Omega,a,b}$ is the law of a random curve in $\overline{\Omega}$ connecting two boundary points a and b of a simply connected domain Ω. Choose some consistent parameterization for such curves so that they are parametrized by $t \in [0,\infty)$. Now we use *Schramm's principle* (which appeared in the seminal paper [8] by Schramm, see e.g. the discussion about LERW in the introduction of that paper. It is formulated in the following way in [10].) and we demand that $(\mu_{\Omega,a,b})$ satisfies the following two requirements:

(CI) Conformal invariance: For any triplet (Ω,a,b) and any conformal mapping $\phi : \Omega \to \mathbb{C}$, it holds that $\phi\mu_{\Omega,a,b} = \mu_{\phi(\Omega),\phi(a),\phi(b)}$.

(DMP) Domain Markov property: Suppose we are given $\gamma[0,t]$, $t > 0$. The conditional law of $\gamma(t+s)$ given $\gamma[0,t]$ is the same as the law of $\gamma(s)$ in the slit domain $(\Omega \setminus \gamma[0,t], \gamma(t), b)$. That is

$$\mu_{\Omega,a,b}(\,\cdot \mid \gamma[0,t]) = \mu_{\Omega\setminus\gamma[0,t],\gamma(t),b}.$$

First of all CI tells that $\mu_{\Omega,a,b} = \phi\mu_{\mathbb{H},0,\infty}$, where ϕ is a conformal mapping from the triplet $(\mathbb{H},0,\infty)$ to the triplet (Ω,a,b). Note that ϕ is not unique: any $\phi(\lambda\,\cdot), \lambda > 0$ would also do. So for each (Ω,a,b) choose some $\Phi = \phi$.

Now we can restrict to the standard triplet $(\mathbb{H},0,\infty)$. Let H_t be the unbounded component of $\mathbb{H} \setminus \gamma[0,t]$, K_t the complement of H_t in $\mathbb{H}$ and g_t the mapping associated with K_t. The combination of CI and DMP shows that the curve $\tilde{\gamma} : s \mapsto g_t\big(\gamma(t+s)\big) - X_t$ is independent of $\gamma[0,t]$ and is identically distributed to γ. This leads to the fact that X_t has independent and stationary increments. Since K_t, defined by a curve, is growing locally, it has a continuous driving process. All the continuous processes with independent and stationary increments are of the form

$$X_t = \sqrt{\kappa}B_t + \theta t,$$

with some constants $\kappa \geq 0$ and $\theta \in \mathbb{R}$. Here B_t is a standard one-dimensional Brownian motion. Let $\phi_\lambda : z \mapsto \lambda z$. CI with $\phi = \phi_\lambda$ implies that X_t and $\lambda X_{t/\lambda^2}$ have the same law. This shows that $\theta = 0$ and furthermore that the law of the random curve in (Ω,a,b) doesn't depend on the choice of Φ.

Chordal SLE_κ is the law of K_t with the driving process $X_t = \sqrt{\kappa}B_t$. It turns out that K_t is generated by a curve in the sense that there is a curve γ so that $\mathbb{H} \setminus K_t$ is the unbounded component of $\mathbb{H} \setminus \gamma[0,t]$, see [7]. Such γ is called the *trace*. For $\kappa \in (0,4]$ it is a simple curve.

2.2 Strip SLE and the Upper-Half Plane $\mathrm{SLE}_\kappa(\rho)$

It is possible to repeat Schramm's principle for three marked boundary points. A natural domain for three marked points is the infinite strip $S_\pi = \{z \in \mathbb{C} : 0 < \mathrm{Im}(z) < \pi\}$. The marked points are now 0, $-\infty$ and $+\infty$.

We can continue in the same way as in the case of the upper half-plane. For a family of hulls $(K_t)_{t\geq 0}$ on the strip S_π, let $g_t^{S_\pi}$ be a conformal mapping from $S_\pi \setminus K_t$ onto S_π normalized by $g_t^{S_\pi}(z) = z \pm const. + o(1)$ as $z \to \pm\infty$. We can reparameterize such that

$g_t^{S_\pi}(z) = z \pm t + o(1)$ as $z \to \pm\infty$. The strip Loewner equation is

$$\partial_t g_t^{S_\pi}(z) = \coth\left(\frac{g_t^{S_\pi}(z) - X_t}{2}\right). \tag{2}$$

We can formulate the conformal invariance and the domain Markov property for three marked points by adding a third point c which behaves the same way as b. As in the two point case we can show that the collection of probability measures $(\mu_{\Omega,a,b,c})$ has properties CI and DMP if and only if the driving process of the random curve of $\mu_{S_\pi,0,\infty,-\infty}$ is of the form

$$X_t = \sqrt{\kappa} B_t + \theta t.$$

Now we don't have any conformal mappings other than the identity map preserving $(S_\pi, 0, -\infty, +\infty)$. So in general, θ doesn't need to vanish. Hence the strip SLEs are a family of probability measures parameterized by two real parameters. See also [9].

The infinite strip S_π can be mapped to the upper half-plane by mappings of the form $\phi : z \mapsto \alpha e^{\pm z} + \beta$ where $\alpha, \beta \in \mathbb{R}$ and the sign of α is such that $i\pi/2$ is mapped to the upper half-plane. Choose α and β so that the marked points are mapped in the following way: 0 to $x \in \mathbb{R}$ and one of $-\infty$ or $+\infty$ to ∞ and the other to $y \in \mathbb{R}$. The strip SLE is mapped to a random curve of the upper half-plane by defining $\widehat{K}_t = \phi(K_t)$ which is a collection of hulls of $\mathbb{H}$ parametrized by the "strip capacity". After a time change to the upper half-plane capacity, the half-plane mappings g_t related to these hulls satisfy the half-plane Loewner equation (1) with the driving process defined through the Itô differential equation

$$\mathrm{d}X_t = \sqrt{\kappa}\mathrm{d}B_t + \frac{\rho \mathrm{d}t}{X_t - Y_t}, \tag{3}$$

where $Y_t = g_t(y)$. For details of this coordinate change and time change see [9].

The process $(X_t - Y_t)/\sqrt{\kappa}$ is, in fact, a Bessel process. The parameter ρ depends on θ and κ through

$$\rho = \mp\theta + \frac{\kappa - 6}{2}, \tag{4}$$

where the sign depends on which of the points $-\infty$ or $+\infty$ was mapped to ∞, so that the sign in (4) is opposite to the sign of $\pm\infty$ and the one in front of z in the definition of ϕ, as indicated by the notation. The law of K_t of the above driving process is called $\mathrm{SLE}_\kappa(\rho)$. It was first introduced in [5] and the heuristic motivation given in that paper is essentially the same as one given above based on CI and DMP.

This description works until the stopping time

$$\tau = \inf\{t \geq 0 : |Y_s - X_s| \to 0 \text{ as } s \nearrow t\}. \tag{5}$$

For the strip SLE this is the time when the curve disconnects $-\infty$ from $+\infty$ that is the curve hits $i\pi + \mathbb{R}$. After this the strip SLE can't be continued in any straightforward way. For the upper half-plane SLE τ is the time when the curve disconnects y from ∞ (for $\kappa > 4$) or the curve hits y (for $\kappa \leq 4$). After time τ the upper half-plane SLE can be continued, at least for a range of values of the parameters.

$\mathrm{SLE}_\kappa(\rho)$ are important since they are the random curves of the upper half-plane that depend on three marked points and satisfy Schramm's principle. And especially important is the case $\rho = \kappa - 6$ since that is the coordinate transformation of chordal SLE under a

Möbius map taking the points 0 and ∞ to two points x and y on the real line. This can be seen from (4): The SLE on the strip S_π with parameter values κ and θ is related to the one with values κ and $-\theta$ by mirroring with respect to the imaginary axis. In the upper half-plane this corresponds to changing from κ and ρ to the parameter values κ and $-\rho+\kappa-6$ by (4). This transformation can be completed by making one more mirroring in the upper half-plane. The result is a conformal self-mapping of the upper half-plane. For example, $m \circ \exp \circ m \circ \log(z) = -1/z$, where $m(z) = -\overline{z}$ is the mirroring map.

Since for $\kappa \in (0,8)$ the chordal SLE avoids almost surely a given point in $\overline{\mathbb{H}} \setminus \{0\}$, it avoids especially the point that is mapped to ∞. From this it follows that the image of the full trace $\gamma(0,\infty)$ under the Möbius map is a bounded set. Hence considering $\mathrm{SLE}_\kappa(\kappa-6)$ makes it possible to study the properties of the full trace of chordal SLE_κ. They really agree when $\kappa \le 4$ and the curves are simple. But also for $\kappa > 4$, it can be useful to consider the part until the disconnection time τ, which is also the time until the two curves agree.

It is also possible to see from the equation (4) that if the interface of an Ising type model with $(+,-,\text{free})$-boundary condition has a scaling limit that is $\mathrm{SLE}_\kappa(\rho)$ then it has to be $\theta = 0$ and $\rho = (\kappa-6)/2$. Namely, the law of the curve has to be invariant under the mirroring map, since the setup is invariant under the mapping that changes signs of all spins. This special case is also called dipolar SLE, see [1].

3 Stationarity and some Example Calculations

3.1 Stationarity of SLE

Now we are ready to present the key idea of this paper. We will take a random conformal mapping and require that its law is invariant under SLE flow. Such a random conformal mapping is said to have *stationary* law. Based on this invariance we can derive equations satisfied by quantities related to SLE.

Let $x, y \in \mathbb{R}$, $x \neq y$. Consider $\mathrm{SLE}_\kappa(\rho)$ so that $X_0 = x$ and $Y_0 = y$, X_t is the driving process, Y_t is as above, and g_t is the Loewner map. Let $\phi_t(z) = \alpha_t z + \beta_t$ be the transformation that maps the points x and y to the points X_t and Y_t. We require that

$$\begin{cases} \phi_t(x) = X_t, \\ \phi_t(y) = Y_t. \end{cases} \tag{6}$$

From these equations we solve for the processes α_t and β_t.

Consider a random conformal map $\tilde{G} : \mathbb{H} \setminus \tilde{K} \to \mathbb{H}$ that is normalized by $\tilde{G}(z) = z + o(1)$ at infinity, and independent from the SLE given by X_t and preserved by the SLE flow in the following sense: the mapping

$$G_t = \phi_t \circ \tilde{G} \circ \phi_t^{-1} \circ g_t \tag{7}$$

has the same law as $\tilde{G}$. This property is schematically illustrated in Fig. 1. The following theorem states that the mapping $\tilde{G} = \tilde{g}_{\tilde{\tau}}$ has this property where $\tilde{g}_t$ is $\mathrm{SLE}_\kappa(\rho)$ and independent of g_t, and $\tilde{\tau}$ is the stopping time defined analogously as in (5).

Theorem 1 *Let the pair (g_t, τ) be $\mathrm{SLE}_\kappa(\rho)$ and the stopping time of (5), and let $(\tilde{g}, \tilde{\tau})$ be an independent copy of them. If $\rho < (\kappa-4)/2$ then $\tau < \infty$ a.s. and hence g_τ is well-defined.*

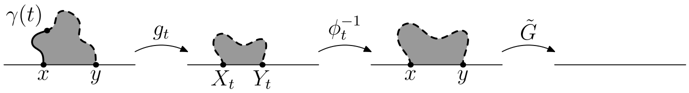

Fig. 1 The law of $\tilde{G}$ is stationary in the following sense: if the law of the hull in the third picture is taken according to the law of $\tilde{G}$ and if an independent piece of SLE is added as in the first picture, then the law of this modified hull is the same as the first one

Furthermore, if ϕ_t is as above, then $\tilde{G} = \tilde{g}_{\tilde{\tau}}$ and

$$G_t = \begin{cases} \phi_t \circ \tilde{G} \circ \phi_t^{-1} \circ g_t & \text{on } \{\tau > t\} \\ g_\tau & \text{on } \{\tau \le t\} \end{cases} \tag{8}$$

are identically distributed.

Proof The argument we present here is basically that $\mathrm{SLE}_\kappa(\rho)$ satisfies Schramm's principle for three marked points. Since we didn't provide the details above, it is worth writing down. The theorem and the proof relies on the Markov property and the Brownian scaling of Bessel processes.

Assume that $x < y$. The other case can be done symmetrically. Write the Bessel stochastic differential equation in a bit non-standard way as

$$\mathrm{d}Z_t = \sqrt{\kappa}\mathrm{d}B_t + (\rho+2)\frac{\mathrm{d}t}{Z_t}. \tag{9}$$

Let Z_t and $\tilde{Z}_t$ be the solutions of (9) for two independent Brownian motions and with the initial condition $Z_0 = \tilde{Z}_0 = y - x$. Now the driving process X_t is defined through the equations

$$Y_t = Y_0 + \int_0^t \frac{2\mathrm{d}s}{Z_s},$$
$$X_t = Y_t - Z_t.$$

In the same way using $\tilde{Z}_t$ instead of Z_t define $\tilde{X}_t$ and $\tilde{Y}_t$. The stopping time τ can be written as

$$\tau = \inf\{t \ge 0 : Z_s \to 0 \text{ as } s \nearrow t\}$$

and $\tilde{\tau}$ can be written using $\tilde{Z}_t$.

The first claim follows from the fact that Z_t is a scaled version of a Bessel process defined using the standard normalization, with the index

$$\nu = 2\frac{\rho+2}{\kappa}.$$

A standard fact is that a Bessel process will hit 0 if and only if $\nu < 1$, see Example 6.5.3 of [2].

The mapping $\phi_t \circ \tilde{g}_s \circ \phi_t^{-1}$ satisfies the normalization

$$\phi_t \circ \tilde{g}_s \circ \phi_t^{-1}(z) = z + \frac{2\alpha_t^2 s}{z} + \cdots$$

and the family of mappings $\hat{g}_s = \phi_t \circ \tilde{g}_{s/\alpha_t^2} \circ \phi_t^{-1}$ satisfies the Loewner equation with the driving process

$$\begin{aligned}\hat{X}_s = \alpha_t \tilde{X}_{s/\alpha_t^2} + \beta_t &= \alpha_t \left(\tilde{Y}_0 - \tilde{Z}_{s/\alpha_t^2} + \int_0^{s/\alpha_t^2} \frac{2\mathrm{d}u}{\tilde{Z}_u} \right) + \beta_t \\ &= Y_t - \alpha_t \tilde{Z}_{s/\alpha_t^2} + \alpha_t \int_0^{s/\alpha_t^2} \frac{2\mathrm{d}u}{\tilde{Z}_u}.\end{aligned}$$

Since the second and third term satisfy the Brownian scaling we can write

$$\hat{X}_s = Y_t - \hat{Z}_s + \int_0^s \frac{2\mathrm{d}u}{\hat{Z}_u}$$

where $\hat{Z}_s$ is a solution of the Bessel SDE (9) with the initial value $\hat{Z}_0 = \alpha_t \tilde{Z}_0 = Y_t - X_t$. Hence the process defined as

$$\begin{cases} Z_s & \text{when } s \le t \\ \hat{Z}_{s-t} & \text{when } s > t \end{cases}$$

is distributed as the process Z_s by the Markov property of Bessel processes and $\hat{g}_s \circ g_t$ is distributed as g_{t+s}. Let σ be the stopping time for $\hat{Z}_s$ hitting 0 as $s \nearrow \sigma$. Then $\sigma = \tilde{\tau}\alpha_t^2$. And hence on $\{\tau > t\}$ the mapping $\phi_t \circ \tilde{G} \circ \phi_t^{-1} \circ g_t$ has the same law as g_τ. On $\{\tau \le t\}$ the statement follows immediately. □

Note that by the strong Markov property t could be replaced by a stopping time in the previous theorem. In the previous proof, Feller's test [2] gave the sharp condition for that $\tau < \infty$ a.s. By considering third point $W_t = g_t(w)$, $w \in \mathbb{R}$, such that $x < y < w$, and applying the same test to $\hat{W} = (W_t - X_t)/(Y_t - X_t)$ (or rather its time change) it is possible to conclude the condition under which $\gamma(\tau) = y$ a.s., is $\rho \le (\kappa - 8)/2$.

For small t, the event $\{\tau \le t\}$ has exponentially small probability. To see this we need to consider only the diffusion term ($\mathrm{d}B_t$) of (9) and we need to note that the probability that a Brownian motion started from $y - x$ comes near 0 in the time interval $[0, t]$ is exponentially small in $1/t$. By this property we need basically just care about the first case of (8). Actually we will use the stationarity to calculate the distribution of τ. See (29) below.

Write the expansion of g_t as

$$g_t(z) = z + \frac{a_1(t)}{z} + \frac{a_2(t)}{z^2} + \cdots. \tag{10}$$

We call *SLE data* the collection of random variables

$$X_t, Y_t, a_1(t), a_2(t), \ldots \tag{11}$$

SLE data carries all the information about g_t and the law of g_s, $s > t$. The coefficient $a_1(t) = 2t$ and the higher coefficient are definite integrals of polynomials on the lower coefficients and X_t. So in principle, they could be calculated. On the stopping time τ we have $X_t - Y_t \to 0$ as $t \nearrow \tau$ and then the SLE data simplifies to $a_k(\tau)$, $k \in \mathbb{N}$. Note that also $a_1(\tau) = 2\tau$ is random.

During the rest of this paper we will present some examples how to use the stationarity to calculate SLE data related quantities, like the moments $\mathbb{E}[\prod a_{k_i}(\tau)]$.

It should be stressed, that the expected value $\mathbb{E}[\prod a_{k_i}(\tau)]$ exists only for a certain range of the parameters κ, ρ. For example, when $\rho = \kappa - 6$, for any $\kappa < 8$, $\tau < \infty$ a.s. and g_τ is well-defined, but $\mathbb{E}[\prod |a_{k_i}(\tau)|] < \infty$ only for $0 \le \kappa < \kappa_0(k_1, \dots, k_n)$ where $\kappa_0(k_1, \dots, k_n) \to 0$ as a natural degree of $(k_1, \dots, k_n)$ grows. This will be commented more in the end of Sect. 3.5.

3.2 Basic Equations for the Coefficients of $\tilde{G}$

In this section we derive the equation describing the flow of $(\tilde{a}_k)$ under the flow (8). Use the expansion

$$\tilde{G}(z) = z + \frac{\tilde{a}_1}{z} + \frac{\tilde{a}_2}{z^2} + \cdots$$

to write the expansion of G_t of (8)

$$\begin{aligned} G_t(z) &= \alpha_t \tilde{G}\left(\frac{g_t(z) - \beta_t}{\alpha_t}\right) + \beta_t \\ &= g_t(z) + \frac{\tilde{a}_1 \alpha_t^2}{g_t(z) - \beta_t} + \frac{\tilde{a}_2 \alpha_t^3}{(g_t(z) - \beta_t)^2} + \cdots . \end{aligned} \tag{12}$$

So to get the Itô differential of the expansion we need to calculate Itô differential of $g_t(z)$ and expressions of type $\alpha_t^{n+1}/(g_t(z) - \beta_t)^n$ at time $t = 0$.

Let's simplify the setup: let $\sigma \in \{-1, 1\}$ and $x = \sigma$ and $y = -\sigma$. Note we can always transform the above setup to this simplified setup with scaling and translation. Now

$$\mathrm{d}g_t(z) \underset{t=0}{=} \frac{2\mathrm{d}t}{z - \sigma} = \left\{\frac{2}{z} + \sigma\frac{2}{z^2} + \frac{2}{z^3} + \sigma\frac{2}{z^4} + \cdots\right\}\mathrm{d}t \tag{13}$$

and after a short calculation we find that

$$\begin{aligned} \mathrm{d}\frac{\alpha_t^{n+1}}{(g_t(z) - \beta_t)^n} \underset{t=0}{=} & \left\{\left[(n+1)\frac{\rho+2}{4} + n(n+1)\frac{\kappa}{8}\right]\frac{1}{z^n}\right. \\ & + \sigma\left[n\frac{\rho-2}{4} + n(n+1)\frac{\kappa}{4}\right]\frac{1}{z^{n+1}} + \left[-2n + n(n+1)\frac{\kappa}{8}\right]\frac{1}{z^{n+2}} \\ & \left. - \sigma\,\frac{2n}{z^{n+3}} - \frac{2n}{z^{n+4}} - \cdots\right\}\mathrm{d}t + \left\{\sigma(n+1)\frac{1}{z^n} + n\frac{1}{z^{n+1}}\right\}\mathrm{d}B_t. \end{aligned} \tag{14}$$

Using the notation $G_t(z) = z + a_1(t)z^{-1} + a_2(t)z^{-2} + \cdots$ and combining last two Itô differentials with (12) we finally get

$$\begin{aligned} \mathrm{d}a_n(t) \underset{t=0}{=} & \left\{\frac{1}{8}(n+1)(\kappa n + 2\rho + 4)\tilde{a}_n + \sigma\frac{1}{4}(n-1)(\kappa n + \rho - 2)\tilde{a}_{n-1}\right. \\ & \left. + \frac{1}{8}(n-2)(\kappa(n-1) - 16)\tilde{a}_{n-2} - \sum_{k=1}^{n-3} 2k\sigma^{n-k}\tilde{a}_k + 2\sigma^{n+1}\right\}\mathrm{d}t \\ & + \frac{\sqrt{\kappa}}{2}\left\{\sigma(n+1)\tilde{a}_n + (n-1)\tilde{a}_{n-1}\right\}\mathrm{d}B_t. \end{aligned} \tag{15}$$

From now on we will consider only $t = 0$ and therefore we can identify $\tilde{a}_n$ and a_n. Write in short

$$\mathrm{d}a_n = \left(c_{n,0} + \sum_{k=1}^{n} c_{n,k} a_k \right) \mathrm{d}t + \left(d_{n,n-1} a_{n-1} + d_{n,n} a_n \right) \mathrm{d}B_t. \tag{16}$$

These expressions are linear in variables (a_k) and hierarchical in the sense that the Itô differential of a_n involves only terms a_k for $k \leq n$. This is really the reason why this method is useful.

3.3 Stationarity for the Inverse Mapping

Similar argument can be made for the inverse mapping $\tilde{F} = \tilde{G}^{-1} : \mathbb{H} \to \mathbb{H} \setminus \tilde{K}$. For the inverse mapping f_t of g_t the Loewner equation is

$$\partial_t f_t(z) = -f_t'(z) \frac{2}{z - X_t}. \tag{17}$$

Let $\tilde{F}$ be a random conformal mapping that is preserved by SLE flow of f_t in the following sense: the mapping

$$F_t = f_t \circ \phi_t \circ \tilde{F} \circ \phi_t^{-1} \tag{18}$$

has the same law as $\tilde{F}$.

Now $F_t(z) = f_t(\alpha_t \tilde{F}((z - \beta_t)/\alpha_t) + \beta_t)$ and therefore

$$\mathrm{d}F_t(z) \underset{t=0}{=} -\frac{2\mathrm{d}t}{\tilde{F}(z) - \sigma} + \mathrm{d}\left(\alpha_t \tilde{F}\left(\frac{z - \beta_t}{\alpha_t} \right) + \beta_t \right). \tag{19}$$

If $\tilde{F}(z) = z + \tilde{b}_1 z^{-1} + \tilde{b}_2 z^{-2} + \cdots$ and $F_t(z) = z + b_1(t) z^{-1} + b_2(t) z^{-2} + \cdots$, we get expression for $\mathrm{d}b_n(t = 0)$ in terms of $\tilde{b}_m$ similarly as in the case of $\tilde{G}$. But now the expressions are not linear in $\tilde{b}_m$. For this reason we won't consider this setup.

3.4 The Reversibility of SLE with Moments

The reversibility of SLE is the following property: let γ be chordal SLE from 0 to ∞. Then γ and $-1/\gamma$ appropriately parameterized have the same law. In terms of $\mathrm{SLE}_\kappa(\kappa - 6)$ this can be stated as $\mathrm{SLE}_\kappa(\kappa - 6)$ from x to y and $\mathrm{SLE}_\kappa(\kappa - 6)$ from y to x appropriately parameterized have the same law. Especially this means that the hulls of the full traces have to have the same law.

Consider now $x = -1$ and $y = 1$. Start $\mathrm{SLE}_\kappa(\kappa - 6)$ from x and denote by τ_- the hitting time of y and let the conformal map be $g^-_{\tau_-}(z) = z + a_1^- z^{-1} + a_2^- z^{-2} + \cdots$. In the same way start $\mathrm{SLE}_\kappa(\kappa - 6)$ from y and denote by τ_+ the hitting time of x and let the conformal map be $g^+_{\tau_+}(z) = z + a_1^+ z^{-1} + a_2^+ z^{-2} + \cdots$. The reversibility can be formulated using the coefficient $a_n^\pm$: for any $n \in \mathbb{N}$ and $l_1, \ldots, l_n \in \mathbb{N}$, $l_1 < l_2 < \cdots < l_n$

$$(a^-_{l_1}, a^-_{l_2}, \ldots, a^-_{l_n}) \overset{\mathcal{L}}{=} (a^+_{l_1}, a^+_{l_2}, \ldots, a^+_{l_n}),$$

i.e. they have the same law.

Let $m(z) = -\overline{z}$. This map is the mirror map that changes x with y and therefore

$$m \circ g^-_{\tau_-} \circ m \stackrel{\mathcal{L}}{=} g^+_{\tau_+}.$$

On the other hand for any $g(z) = z + a_1 z^{-1} + a_2 z^{-2} + \cdots$ with real a_m, $m \in \mathbb{N}$, we have

$$\begin{aligned} m \circ g \circ m(z) &= m(g(-\overline{z})) = m\left(-\overline{z} - \frac{a_1}{\overline{z}} + \frac{a_2}{\overline{z}^2} - \frac{a_3}{\overline{z}^3} + \cdots\right) \\ &= z + \frac{a_1}{z} - \frac{a_2}{z^2} + \frac{a_3}{z^3} + \cdots . \end{aligned}$$

In words, the even coefficients change sign under the mirror map m. This shows that the reversibility is equivalent to

$$(a^-_{l_1}, a^-_{l_2}, \ldots, a^-_{l_n}) \stackrel{\mathcal{L}}{=} ((-1)^{l_1+1} a^-_{l_1}, (-1)^{l_2+1} a^-_{l_2}, \ldots, (-1)^{l_n+1} a^-_{l_n}) \tag{20}$$

which a nice way to give a concrete formulation for the reversibility.

Let $n \in \mathbb{N}$ and $(k_1, \ldots, k_n) \in \{0, 1, 2, \ldots\}^n$. If the reversibility holds then by (20) each a^-_j with even j collects a minus sign to the following moment and hence

$$\mathbb{E}\left[\prod_{j=1}^{n} (a^-_j)^{k_j}\right] = (-1)^{\sum_{1 \le i \le n/2} k_{2i}} \, \mathbb{E}\left[\prod_{j=1}^{n} (a^-_j)^{k_j}\right] \tag{21}$$

which therefore should vanish when $\sum_{1 \le i \le n/2} k_{2i}$ is odd. In fact, if every moment existed, one strategy in proving the reversibility, at least in the case $\kappa \in (0, 4]$, could be showing that these odd moments vanish and showing that the moments determine the distribution.

3.5 General Expression for Moments

To work out equations for expected values of the type in (21) we use the following notation: fix $n \in \mathbb{N}$ and $(k_1, \ldots, k_n) \in \{0, 1, 2, \ldots\}^n$ and let

$$\Pi = \Pi(k_1, k_2, \ldots, k_n) = a_1^{k_1} a_2^{k_2} \cdot \cdots \cdot a_n^{k_n}$$

and for $i \in \{1, \ldots, n\}$

$$\begin{aligned} \Pi^i(k_1, k_2, \ldots, k_n) &= \Pi(k_1, \ldots, k_{i-1}, k_i + 1, k_{i+1}, \ldots, k_n), \\ \Pi_i(k_1, k_2, \ldots, k_n) &= \Pi(k_1, \ldots, k_{i-1}, k_i - 1, k_{i+1}, \ldots, k_n). \end{aligned}$$

Here $\Pi = 0$ with negative arguments. Define similarly $\Pi^{j_1, \ldots, j_m}_{i_1, \ldots, i_l}$. Further $\Pi^0 = \Pi$. Since we are looking for the stationary $\tilde{G}$ we require that the expectation of the drift of Π vanishes. So for a while we will manipulate the expression of $\mathrm{d}\Pi$.

Using this notation and the notation of equation (16) we find that

$$\mathrm{d}\Pi = \sum_i k_i \Pi_i \mathrm{d}a_i + \frac{1}{2} \sum_{i,j} k_i (k_j - \delta_{ij}) \Pi_{i,j} \mathrm{d}a_i \mathrm{d}a_j$$

$$= \cdots = \Bigg\{ \bigg[\frac{1}{2} \sum_i k_i d_{ii} \bigg(2\frac{c_{ii}}{d_{ii}} - d_{ii} + \sum_j k_j d_{jj} \bigg) \bigg] \Pi$$
$$+ \sum_{i>1} k_i d_{i,i-1} \bigg(\frac{c_{i,i-1}}{d_{i,i-1}} - d_{ii} + \sum_j k_j d_{jj} \bigg) \Pi_i^{i-1}$$
$$+ \frac{1}{2} \sum_{i,j>1} k_i (k_j - \delta_{ij}) d_{i,i-1} d_{j,j-1} \Pi_{i,j}^{i-1,j-1}$$
$$+ \sum_i k_i \sum_{l=0}^{i-2} c_{il} \Pi_i^l \Bigg\} \mathrm{d}t + \{ \quad \} \mathrm{d}B_t. \tag{22}$$

Note that the following expressions are independent of the summation index i for any κ and ρ

$$2\frac{c_{ii}}{d_{ii}} - d_{ii} = \frac{2\rho + 4 - \kappa}{2\sqrt{\kappa}},$$
$$\frac{c_{i,i-1}}{d_{i,i-1}} - d_{ii} = \frac{\rho - 2 - \kappa}{2\sqrt{\kappa}}.$$

Next we write that $\sum_i k_i d_{ii} = \sqrt{\kappa}/2 \sum_i k_i (i+1) = \sqrt{\kappa} N$, which defines the degree

$$N = \frac{1}{2} \sum_i k_i (i+1) \tag{23}$$

of a moment Π. Plugging this and the values of c_{ij} and d_{ij} we get that

$$\mathrm{d}\Pi = \Bigg\{ \frac{1}{4} N[(2\rho + 4 - \kappa) + 2N\kappa]\Pi + \sigma \frac{1}{4}[(\rho - 2 - \kappa) + 2N\kappa] \sum_i k_i (i-1) \Pi_i^{i-1}$$
$$+ \frac{\kappa}{8} \sum_{i,j} k_i (k_j - \delta_{ij})(i-1)(j-1) \Pi_{i,j}^{i-1,j-1}$$
$$- 2 \sum_i k_i \sum_{l=1}^{i-2} \sigma^{i-l} l \, \Pi_i^l + 2 \sum_i k_i \sigma^{i+1} \Pi_i^0 \Bigg\} \mathrm{d}t + \{ \quad \} \mathrm{d}B_t. \tag{24}$$

For $\rho = \kappa - 6$ the above brackets are $2\rho + 4 - \kappa + 2N\kappa = (2N+1)\kappa - 8$ and $\rho - 2 - \kappa + 2N\kappa = 2N\kappa - 8$. Let's use this value of ρ for a while.

Now we analyze the degree N. First of all

$$N = \frac{1}{2} \sum_i k_{2i-1} \cdot 2i + \frac{1}{2} \sum_i k_{2i} \cdot (2i+1)$$
$$= \sum_i (k_{2i-1} + k_{2i}) i + \frac{1}{2} \sum_i k_{2i}.$$

So N is either a half-integer or an integer depending whether $\sum_i k_{2i}$ is odd or even. So for the reversibility we would like to show that $\mathbb{E}[\Pi] = 0$ when N is a half-integer. Next we note that the drift in the equation (24) decomposes into $A + \sigma B$ where A and B don't depend (directly) on σ and all the half-integer moments are put in the other one and the integer moments on the other.

Under the reversibility $\mathbb{E}[\Pi]=0$ when N is a half-integer, then for N an integer we would have

$$\mathbb{E}[\Pi(k_1,\dots,k_n)]=\frac{p_{k_1,\dots,k_n}(\kappa)}{(8-3\kappa)(8-5\kappa)\cdot\ldots\cdot(8-(2N+1)\kappa)}, \tag{25}$$

where $p_{k_1,\dots,k_n}$ is a polynomial with highest degree $\tilde{N}=1/2\sum_i k_i(i-1)$. The denominator follows from the fact that as we recursively solve $\mathbb{E}[\Pi]$ from (24) by demanding that the drift vanishes, the factor in front of the moment with the largest degree is $1/4N[(2N+1)\kappa-8]$. Similarly κ can enter the numerator only through the term $\Pi_{i,j}^{i-1,j-1}$ (this argument requires more care though). $\tilde{N}$ is the number of steps from $\Pi(k_1,\dots,k_n)$ to $\Pi(k_1',0,0,\dots,0)$ by lowering two powers with $\Pi_{i,j}^{i-1,j-1}$.

Equation (25) can be interpreted so that the expected value $\Pi(k_1,\dots,k_n)$ exists for small κ as long as the right-hand side is finite. So we can read from this general form that the expected value $\Pi(k_1,\dots,k_n)$ exists for $\kappa\in\big(0,8/(2N+1)\big)$. This result is proven in Appendix A.1 of [3]. The result therein includes both the half-integer and the integer moments.

3.6 Calculating Moments a_1^n, $a_1^n a_2^m$ and so on

In this section, we study only the case $\rho=\kappa-6$. We will show how to actually calculate moments, i.e. expected values of SLE data. Let's calculate Itô differential

$$\begin{aligned}
\mathrm{d}(a_1^n)&=na_1^{n-1}\mathrm{d}a_1+\frac{1}{2}n(n-1)a_1^{n-2}(\mathrm{d}a_1)^2\\
&=\left[na_1^{n-1}\left(2+\frac{3\kappa-8}{4}a_1\right)+\frac{1}{2}n(n-1)a_1^{n-2}\cdot\kappa a_1^2\right]\mathrm{d}t+(\quad)\mathrm{d}B_t\\
&=n\left[2a_1^{n-1}+\frac{(2n+1)\kappa-8}{4}a_1^n\right]\mathrm{d}t+(\quad)\mathrm{d}B_t.
\end{aligned}$$

Then we demand that expectation of the drift is zero. This gives

$$\mathbb{E}[a_1^n]=\frac{8\mathbb{E}[a_1^{n-1}]}{8-(2n+1)\kappa}=\cdots=\frac{8^n}{(8-3\kappa)(8-5\kappa)\cdot\cdots\cdot(8-(2n+1)\kappa)}$$

since $\mathbb{E}[a_1^0]=1$. This is true for $x=\sigma$ and $y=-\sigma$. For general $x,y\in\mathbb{R}$, use a suitable Möbius transformation to get

$$\mathbb{E}[a_1^n]=\frac{2^n(x-y)^{2n}}{(8-3\kappa)(8-5\kappa)\cdot\cdots\cdot(8-(2n+1)\kappa)}. \tag{26}$$

Similar calculation for $a_1^n a_2^m$, m even, gives

$$\mathbb{E}[a_1^n a_2^m]=\frac{2^{2n+3m}\left(\frac{\kappa}{6}\right)^{m/2}\frac{m!}{(\frac{m}{2})!}}{(8-3\kappa)(8-5\kappa)\cdot\cdots\cdot(8-(2n+3m+1)\kappa)}. \tag{27}$$

The higher moments can be in principle calculated using the recursion we get from (24). The author hasn't been able to completely solve the recursion.

3.7 Density Function of a_1

As stated earlier a_1 is distributed as 2τ where τ is the hitting time of 0 for a Bessel process. Its distribution can be calculated many ways using the Bessel process directly and some of the ways resemble quite much what follows. However the following way to calculate the distribution is worth mentioning.

If the capacity $a_1(t)$ has a density function ν_t then

$$\mathbb{E}_t[f(a_1)] = \int_0^\infty f(x)\nu_t(x)\mathrm{d}x$$

for each sufficiently smooth $f : (0, \infty) \to \mathbb{R}$ with compact support. For such a function the Itô differential is

$$\mathrm{d}f(a_1) = \left[\left(2 + \frac{\kappa + 2\rho + 4}{4} a_1\right) f'(a_1) + \frac{\kappa}{2} a_1^2 f''(a_1)\right] \mathrm{d}t + \sigma a_1 \sqrt{\kappa} f'(a_1)\mathrm{d}B_t.$$

For $\nu_t = \nu$ stationary, the expectation of the drift has to vanish

$$\begin{aligned} 0 &= \mathbb{E}\left[\left(2 + \frac{\kappa + 2\rho + 4}{4} a_1\right) f'(a_1) + \frac{\kappa}{2} a_1^2 f''(a_1)\right] \\ &= \int_0^\infty [p(x) f''(x) + q(x) f'(x)]\nu(x)\mathrm{d}x \end{aligned} \tag{28}$$

where $p(x) = (\kappa/2)\,x^2$ and $q(x) = 2 + ((\kappa + 2\rho + 4)/4)\,x$. Since (28) holds for every f smooth and with compact support, we conclude $-(p(x)\nu(x))' + q(x)\nu(x) = C = \text{const}$. If we assume ν and ν' go zero as $x \to 0$, then $C = 0$.

Now we solve

$$\frac{\nu'(x)}{\nu(x)} = \frac{q(x) - p'(x)}{p(x)} = -\frac{3\kappa - 2\rho - 4}{2\kappa}\frac{1}{x} + \frac{4}{\kappa}\frac{1}{x^2}$$

giving

$$\nu(x) = C_{\kappa,\rho} x^{-\frac{3\kappa - 2\rho - 4}{2\kappa}} e^{-\frac{4}{\kappa}\frac{1}{x}}. \tag{29}$$

Coefficient $C_{\kappa,\rho}$ is determined from $\int_0^\infty \nu(x)\mathrm{d}x = 1$, where the integral converges if and only if the power of x is smaller than -1. For $\rho = \kappa - 6$ this means $\kappa < 8$. This result can be explained as follows: for $\kappa < 8$ the chordal SLE a.s. avoids given point and hence the capacity seen from this point is a.s. finite. Formula (29) can be compared to a formula on p.98 of [11].

The calculation of $a_1(\tau)$ is easier than the other cases, since in any approach it requires keeping track only of $Y_t - X_t$ and not X_t and Y_t separately. The problem is essentially one-dimensional.

4 Conclusions

It was shown how to formulate the stationarity of $\mathrm{SLE}_\kappa(\rho)$ as stationarity of the law of a stopped hull under a SLE induced flow. One of the advances of this approach is that it involves the full SLE trace directly. The full trace is the most interesting object from the statistical physics point of view.

When using the approach to calculate the moments $\mathbb{E}[\prod a_{k_j}]$, the problem is that these expected values only exist for a range of the parameter κ. Hence the approach should be applied in some different way. For example, some other function of the random variables $a_1, a_2, \ldots$ could be taken, say, such as $\mathbb{E}[e^{i\lambda a_1} \prod a_{k_j}]$. As proposed by Stanislav Smirnov, one option is to try to find an alternative interpretation beyond the blowup for the analytic continuations of the moment formulas such as (26) and (27).

Acknowledgements I wish to thank Stanislav Smirnov. This work was started during a joint project. I wish also thank Kalle Kytölä and Antti Kupiainen for useful discussions. Jan Cristina also deserves thanks for reading a part of the paper and for the discussions on writing in English.

References

1. Bauer, M., Bernard, D., Houdayer, J.: Dipolar stochastic Loewner evolutions. J. Stat. Mech. Theory Exp. **2005**(03), P03,001 (2005)
2. Durrett, R.: Stochastic Calculus. Probability and Stochastics Series. CRC, Boca Raton (1996)
3. Kytölä, K.: Conformal field theory methods for variants of Schramm-Loewner evolutions. Doctoral dissertation, University of Helsinki, Faculty of Science, Department of Mathematics and Statistics (2006)
4. Kytölä, K., Kemppainen, A.: SLE local martingales, reversibility and duality. J. Phys. A **39**(46), L657–L666 (2006)
5. Lawler, G., Schramm, O., Werner, W.: Conformal restriction: the chordal case. J. Am. Math. Soc. **16**(4), 917–955 (2003)
6. Lawler, G.F., Schramm, O., Werner, W.: Values of Brownian intersection exponents. I. Half-plane exponents. Acta Math. **187**(2), 237–273 (2001)
7. Rohde, S., Schramm, O.: Basic properties of SLE. Ann. Math. **161**(2), 883–924 (2005)
8. Schramm, O.: Scaling limits of loop-erased random walks and uniform spanning trees. Israel J. Math. **118**, 221–288 (2000)
9. Schramm, O., Wilson, D.B.: SLE coordinate changes. New York J. Math. **11**, 659–669 (2005) (electronic)
10. Smirnov, S.: Towards conformal invariance of 2D lattice models. In: International Congress of Mathematicians, vol. II, pp. 1421–1451. Eur. Math. Soc., Zürich (2006)
11. Yor, M.: Exponential Functionals of Brownian Motion and Related Processes. Springer Finance. Springer, Berlin (2001)
12. Zhan, D.: Reversibility of chordal SLE. Ann. Probab. **36**(4), 1472–1494 (2008)